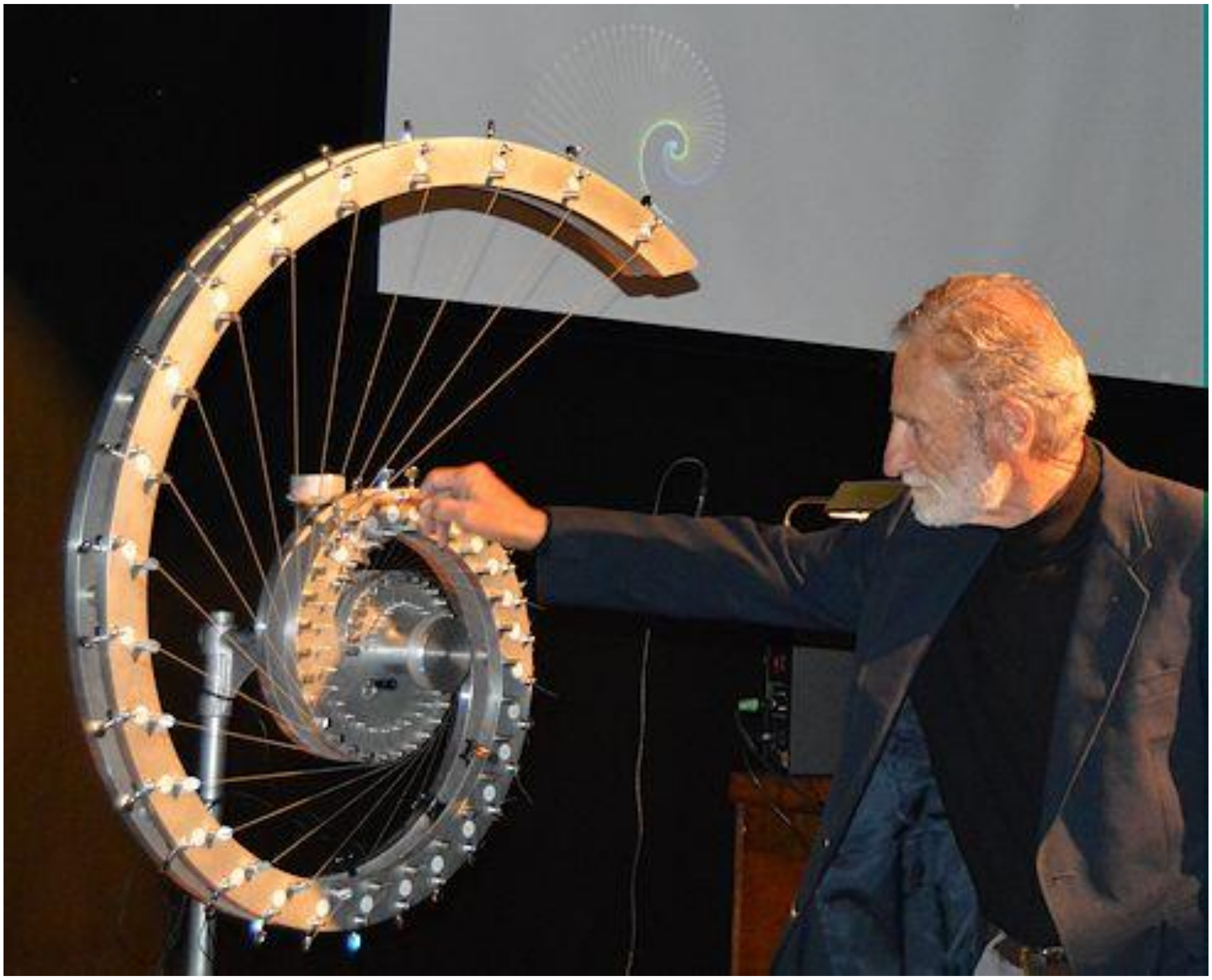
Pitch Formula & AVATAR for Bernoulli Type Instruments

April 6, 2020 1:54 pm Mike Sterling

I write this explanation so that it can help inform D'*A*ddario of my hope to work with them. They know most of this with the exception of some of the mathematics of the AVATAR



The Bernoulli Involute 36 String Instrument w/Mike Sterling pointing out a detail.

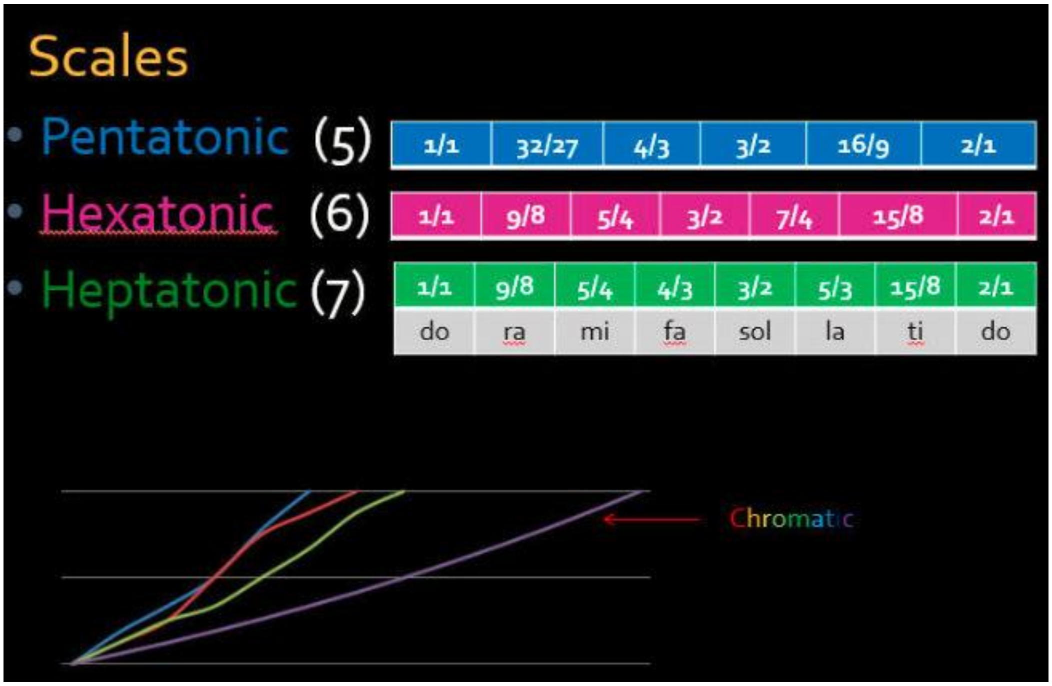
Introduction:

The instruments pictured above was developed after I read a book by Dr. Eli Maor called '***e: The Story of a Number'*** published by Princeton University Press. In Chapter 11, Dr. Maor lets us in on a charming imaginary conversation between J. S. Bach and Johann Bernoulli of the famous Basel Swiss family. The entire book is well written. It, along with other books by Dr. Maor, are best sellers in a genre that is intended to explain our world.

In this Chapter, Bach is having trouble with the accepted musical scale. It has 7 tones with pitch doubling or halving every 8 notes. Bach laments that he cannot transpose his music effectively. The further he gets from the key in which the music was written, the worse it sounds.

Julie Andrews might have handled it, but not if it is transposed too far.

doe, a deer, a female deer
(Re!) ray, a drop of golden sun
(Mi!) me, a name I call myself
(Fa!) far, a long, long way to run
(So!) sew, a needle pulling thread
(La!) la, a note to follow so
(Ti!) tea, a drink with jam and bread



Scales … note how smooth the Chromatic Scale is

The 7 tone scale (Heptatonic – see graphs above) has an unequal distribution of multipliers to get from one note to the next, whereas the Chromatic Scale uses the 12th root of 2 in an orderly easy to understand manner. You can see that the purple line for the chromatic scale is smooth, while the others are not continuous in curvature.

C	C#	D	D#	E	F	F#	G	G#	A	A*	B	C
2^0	$2^{1/12}$	$2^{2/12}$	$2^{3/12}$	$2^{4/12}$	$2^{5/12}$	$2^{6/12}$	$2^{7/12}$	$2^{8/12}$	$2^{9/12}$	$2^{10/12}$	$2^{11/12}$	$2^{12/12}$
130.813	138.591	146.842	155.563	164.814	174.614	184.997	195.998	207.652	220.000	233.082	246.942	261.626
	X			y				X'			y'	

If we have two notes say, C# and E and we want to transpose them moving them to the right and coming to rest on G# and B, we simply move our hands over to G# and B. The pitch of the notes will go up in a uniform way and no harmonic distortion will take place.

They are in exactly the same relation one to another as they were beforehand because the pitch rises by a factor that is uniform as seen in the smooth graph.

Bernoulli proposes using a logarithmic scale of 12 tones with the pitch doubling or halving every 13 notes instead of 8 notes. Note that $2^{12/12} = 2$. Johann is inspired by his eldest brother, Jakob's work on the logarithmic spiral so prevalent in nature. Examples abound …. the Snail, the Sunflower, Spiral Galaxies, the Cochlea in the ear….

This was brought home to me by a picture presented by Dr. Maor of a log scale, or what I prefer to call, the Chromatic Scale. It is graphically depicted on page 132* of Maor's book.

Although being a non-musician and not able to read music quickly, I instantly saw both the problem and the solution.

It dawned on me that I could make instruments that could solve Bach's dilemma and be aesthetically pleasing. The instruments that I foresaw would be instructional in the extreme, leading the way to the connection between music and geometry. I made 2 of the 3 and gave a series of 4 lectures on Geometry and Music.

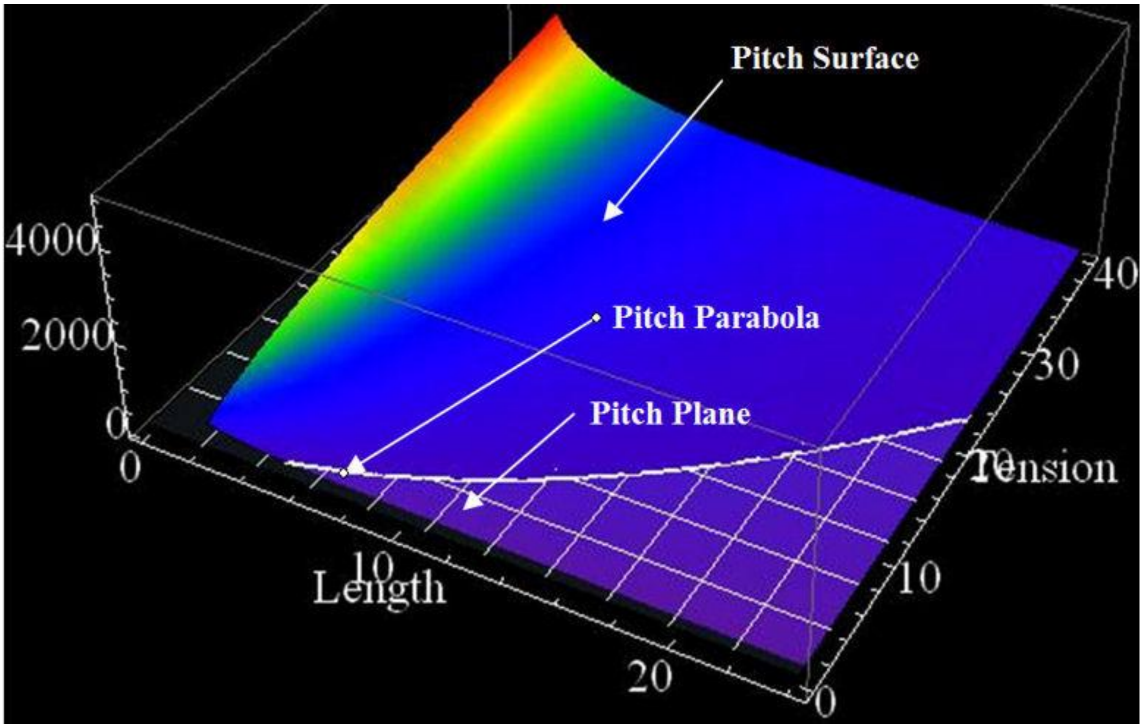
I now understood what 'cents' are and why a 12 tone octave has 1200 cents. It was all clear in an instant. I saw the final solution would include string length and type and of course tension.

I saw that I could build 3 machines, 2 of which are pictured. The third would be a simple design with each string parallel and growing from 3.2" to something more than 24".

If I used the 12th root of 2 as the basis, then a solution would be possible and that it might be fun to hear and make a fully chromatic Instrument. It would use proper unit weight, length, pitch and tension too. Tension would be the great equalizer overcoming any discrepancy in string manufacture or unit weight of the strings.

The first Bernoulli on the garden post. Note the base and the parabolic dish to catch the sound. A single spiral fret is shown embedded in the wood.

Eli and Dalia Maor look over the Bernoulli and a one string tester



The Avatar of a String

The Luther's Equation Inverted

$$\textbf{Pitch} = \frac{1}{2L} Sqrt(\frac{T*386.4)}{uw})$$

Where Pitch = vibrations per second (HZ)

L = String Length in inches

T = Tension (pounds)

UW = Unit Weight in pounds

386.4 comes from gravitational acceleration in inches per second squared. Of course 386.4 changes slightly depending on where we are on earth.

This formula is called the Luthier's Equation. It is used mostly by large string and musical instrument manufacturers for tension estimates. In a 36 string instrument, I would want tension also working with the 12 root of 2.

Muscians are very conscious of tension because it adversely affects the structure of the instrument itself.

They want to make sure the tension is not too high by design. I rearranged terms to concentrate on Pitch. I already have precise string length increasing as the 12^{th} root of 2. Tension is of course handled by the familiar Tuning Machines. It is tied directly to UW, pitch and length. I use guitar tuning machines. They are clearly visible in the above pictures of the Bernoulli Involute. Large harp manufacturers use a very complicated mechanical system that is prone to error. If you have a concert harp, you MUST constantly fiddle with the tension via the Tuning Machines.

The formula defines a 3 dimensional doubly curved surface that is an AVATAR for the real string itself. It is an embodiment of the string, hence the word AVATAR now used in computer science and modelling. It suits what I'm doing. In parametric form it is:

Pitch is a function of (Length, Tension, UW)

Every point on the Pitch Parabola is in tune. See the curving white line.

Lengths are cut to suit. Of course the person selecting the string would cut to a proper length to fit the instrument.

The formula is interesting in that UW is a very small number and it appears in the denominator making the result sensitive to small changes in material weight. We know this intuitively. A string with .040 diameter will normally be used for low pitches, .045 would be lower still, .017 would be used for high pitch. Any attempt at tuning the string using the wrong unit weight will not work well.

I'm interested in the Bernoulli Instruments having the perfect string for the intended pitch. Each string should be made so that UW grows or diminishes like the 12th root of 2. The 12th root of 2 is the ONLY number that will work in this case for the Chromatic Scale. The law of exponents forces the 13th note to have double the pitch of the first note in a chromatic scale. I now do this for length and pitch of course contrary to all harps. They depend upon the tuning mechanisms forcing retuning constantly as atmospherics change the wood or other materials.

The key, therefore, is a precise UW and not diameter. I cannot buy a set of 36 strings off the shelf with the right unit weights.

The unit weight is a function of core and wrap diameter and material used.

How do we make the strings with a given unit weight that grows or diminishes as the 12th root of 2?

I think that D'*A*ddario of Long Island can make them. Their process is perfect for doing so. They have a 3rd generation reputation of being the finest string manufacturer in the world. As a family they go back to the 1600s as string makers.

Here is a video of a Factory tour. Click Here Notice that they make their own core string hexagonal in shape. They then use die sets to make the diameter smaller. They continually monitor the wear. If the die wears too much, they replace it and use the worn die for a larger diameter string.

Finally, the string is wrapped with another string. This added string adheres under tension so that it is almost welded to the core. The pulling and wrapping produce a fused effect.

For the Bernoulli's, I use D'*A*ddario's extensive catalog to search out unit weights that grow smaller going from tenor C to B below High C. Not all unit weights are available so I try to 'fudge' in what is available.

In the Bernoulli Involute case, I use no wrapping for the highest pitched couple of strings. It might be well to always use a wrapped string, if possible. The smallest string length is 3.2". Presently, it produces B below high C. An unwrapped string having the right UW is fine now and should be ok wrapped or unwrapped.

B below High C is a crystal clear pitch on the Bernoulli Involute.

I have briefly worked with Nylon Strings, but I found them a bit dull.

The system I am proposing uses the well-known attributes of the Chromatic Scale. As of now I use UW's that approximate the Chromatic UW's that vary by the 12^{th} root of 2 also.

Of course Tension is adjusted to achieve what I want.   I'd like to complete the precision by using the full power of the Chromatic Scale. That is, I want to eliminate my approximate UWs

The current Bernoullis rarely need tuning unless somebody bumps into the Tuning Machines. Every day I check pitch and it is always good. Due to the construction of the Bernoullis, humidity is not an issue, although the propagation of sound is slightly changed day by day in all instruments. I use D'*A*ddario NYXL Strings which are very stable and quality controlled.

The above picture shows the D'*A*ddario logo with the hexagon representing the core wire and the circle representing the wrapped wire. The die will wear down the vertices of the hexagon producing a fusing of the wire and therefore no slipping of core and wrapping.

There will not be a large market for what I describe, so commercial use of it is difficult to predict. Uses of the idea in small harps is useful. D'*A*ddario makes the best strings for that purpose.

D'*I*ddario would have to make die sets to produce the Chromatic Scale for the present Bernoulli Involute. This is not a deviation from their process.

They will tell me the approximate cost. My need for strings in volume is small. I don't have the time to market the Bernoulli's. If cost is too much for me at the prototype stage, then I will use what I have, which is fine, but I'd like to have the full chromatic sound. I play what I have every day.

The proper chromatic scale machine should be made. According to Dr. Maor it may be unique, I have looked at some small 36 string harps in detail. One typified the difficulty that the designer was having. The natural shape of his instrument had to be overcome. Tension for each string varied all over the place as did length and unit weight due to the shape.   I looked at all the data and understood the problem, but the designer of the harp did not. I have all the data for his harp.

This is where I work …..

Images courtesy of Sandy Lindsay and the Saugeen Times

*(Page 132 is a nice easy number to remember being 12x11. Ramanujan and Hardy's famous Taxi Cab number is $1729 = 12^3+1^3$ and $1729 = 9^3+ 10^3$ which is the smallest number that can be expressed as the sum of 2 cubes in two ways)

Letter of Encouragement from Dr. Eli Maor

Hello Mike,

We have now known each other for nearly ten years, and our friendship has enriched my life in more ways than I can express in words. I therefore allow myself to suggest that you write your involvement with the two *Bernoullis* in a booklet similar to *Walking Euler's Bridge*. I believe the *Bernoullis* are among the few cases in history where a musical instrument was designed specifically according to mathematical principles. You could devote the first part of the booklet to describe, in layman's language, the basic laws of pitch relations, tuning, harmony, etc, and the second part to the more technical details of your invention.

I urge you to give it a serious thought, especially now that we're all being confined to our homes due to the Coronavirus. If you follow this through, I'll be glad to write a brief forward or introduction.

So be well and stay in good health - Eli